Thermodynamic Uncertainty Relations for Coherent Transport

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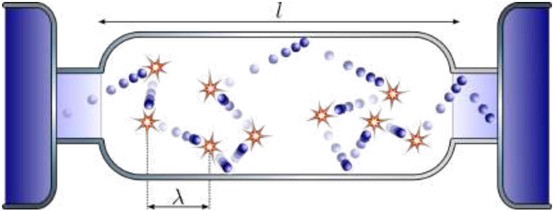
E. Potanina, C. Flindt, M. Moskalets, KB
Phys. Rev. X **11**, 021013 (2021)

KB, T. Hanazato, K. Saito
Phys. Rev. Lett. **120**, 090601 (2018)

Macroscopic Conductors

Macroscopic conductor

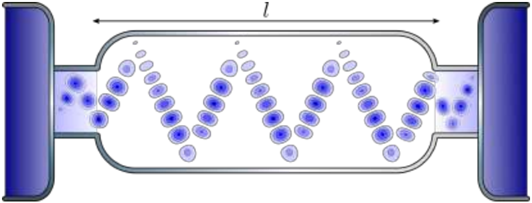
- $l \sim 10^{-2}m \gg \lambda$, $T \sim 10^{3}K$
- Frequent carrier collisions
- Diffusive transport



Coherent Conductors

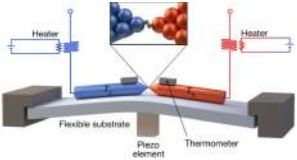
Mesoscopic conductor

- $l \simeq \lambda$
- Non-interacting carriers
- Coherent transport



Atomic junctions

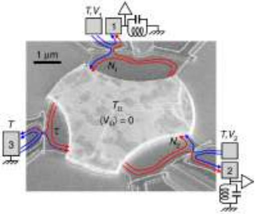
$l \sim 10^{-10} m,\ T \sim 10^{0} K$



O. S. Lumbroso et al., Nature **562** 240 (2018).

Semiconductors

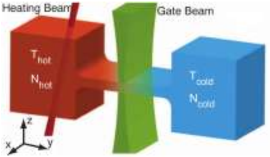
$l \sim 10^{-6} m,\ T \sim 10^{-3} K$



E. Sivre et al., Nat. Com. **10** 5638 (2019).

Cold atoms

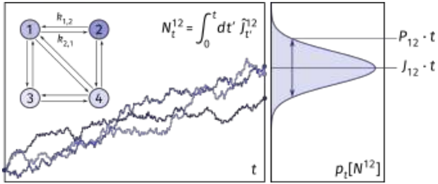
$l \sim 10^{-5} m,\ T \sim 10^{-7} K$



J.-P. Brantut et al., Science **342** 713 (2013).

Thermodynamic Uncertainty Relations - The Beginning

Steady-state time homogeneous Markov-jump process



Empirical current:

$$\hat{J}_t^{yz} \equiv \delta[x_{t-} - y]\delta[x_{t+} - z] - \delta[x_{t-} - z]\delta[x_{t+} - y]$$

$x_t \ldots$ system trajectory

Mean currents and fluctuations:

$$J_{xy} = \lim_{t\to\infty} \frac{1}{t} \int_0^t dt' \, \hat{J}_{t'}^{xy}$$

$$P_{xy} = \lim_{t\to\infty} \frac{1}{t} \int_0^t dt' \int_0^t dt'' \left(\hat{J}_{t'}^{xy} - J_{xy}\right)\left(\hat{J}_{t''}^{xy} - J_{xy}\right)$$

Detailed balance condition:

$$\sigma_{xy}^{\text{env}} = \mathrm{k_B} \ln[k_{xy}/k_{yx}]$$

$\sigma_{xy}^{\text{env}} \ldots$ environmental entropy production

$\mathrm{k_B} \ldots$ Boltzmann's constant

Rate of entropy production:

$$\sigma = \sum_{x,y} \sigma_{xy}^{\text{env}} k_{xy} p_y^{\text{ss}}$$

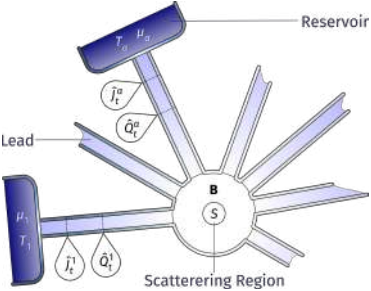
$p_x^{\text{ss}} \ldots$ steady-state occupation of x

Thermodynamic Uncertainty Relation

$$\frac{\sigma P_{xy}}{J_{xy}^2} \geq 2\mathrm{k_B}$$

- A. C. Brato, U. Seifert, PRL **114**, 158101 (2015).
- T. R. Gingrich et al., PRL **116**, 120601 (2016).

Landauer-Büttiker model



B…	magnetic field
$\mu_\alpha \ldots$	chemical potential
$T_\alpha \ldots$	temperature
$\hat{J}_t^\alpha \ldots$	particle current operator
$\hat{Q}_t^\alpha \ldots$	heat current operator
$\langle \cdots \rangle \ldots$	ensemble average

Mean electric and heat currents:

$$J_\alpha = \lim_{t\to\infty} \frac{1}{t} \int_0^t dt' \left\langle \hat{J}_{t'}^\alpha \right\rangle$$

$$Q_\alpha = \lim_{t\to\infty} \frac{1}{t} \int_0^t dt' \left\langle \hat{Q}_{t'}^\alpha \right\rangle$$

Electric current fluctuations:

$$P_\alpha = \lim_{t\to\infty} \frac{1}{t} \int_0^t dt' \int_0^t dt'' \left\langle \left(\hat{J}_{t'}^\alpha - J_\alpha\right)\left(\hat{J}_{t''}^\alpha - J_\alpha\right)\right\rangle$$

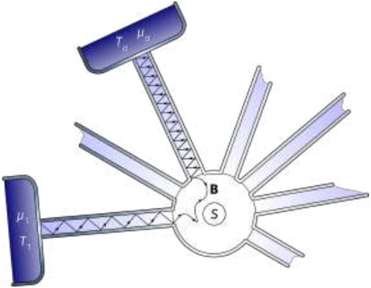
Rate of entropy production:

$$\sigma = -\sum_\alpha Q_\alpha / T_\alpha \geq 0$$

Is there a universal relation between σ, J_α and P_α?

What is the role of quantum effects?

Classical Model



$\hat{J}_t^\alpha \to J^\alpha[\xi_t] \qquad \hat{Q}_t^\alpha \to Q^\alpha[\xi_t]$

$e\ldots$	carrier charge
$h\ldots$	Planck's constant
$E\ldots$	carrier energy
$\mathcal{T}_{E,\mathbf{B}}^{\alpha\beta}\ldots$	transmission coefficients

$u_E^\alpha = \exp[-(E-\mu_\alpha)/k_B T_\alpha]$

Mean electric and heat currents:

$$J_\alpha = \frac{e}{h}\int_0^\infty dE \sum_\beta \mathcal{T}_{E,\mathbf{B}}^{\alpha\beta}\left(u_E^\alpha - u_E^\beta\right)$$

$$Q_\alpha = \frac{1}{h}\int_0^\infty dE \sum_\beta \mathcal{T}_{E,\mathbf{B}}^{\alpha\beta}\left(E-\mu_\alpha\right)\left(u_E^\alpha - u_E^\beta\right)$$

Electric current fluctuations

$$P_\alpha = \frac{e^2}{h}\int_0^\infty dE \sum_{\beta\neq\alpha} \mathcal{T}_{E,\mathbf{B}}^{\alpha\beta}\left(u_E^\alpha + u_E^\beta\right)$$

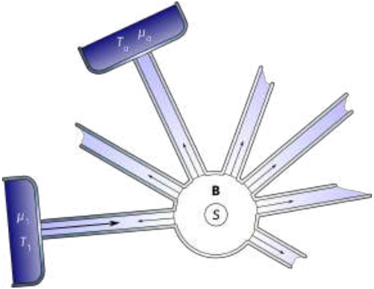
Liouville and TR symmetry:

$$\sum_\alpha \mathcal{T}_{E,\mathbf{B}}^{\alpha\beta} = \sum_\beta \mathcal{T}_{E,\mathbf{B}}^{\alpha\beta} \qquad \mathcal{T}_{E,\mathbf{B}}^{\alpha\beta} = \mathcal{T}_{E,-\mathbf{B}}^{\beta\alpha} \geq 0$$

TUR

$$\frac{\sigma P_\alpha}{J_\alpha^2} \geq \psi k_B \qquad \psi = \begin{cases} 2 & (\mathbf{B}=0) \\ 0.89612 & (\mathbf{B}\neq 0) \end{cases}$$

Quantum Model - Fermionic Carriers - 1d



$$f_E^{\alpha} = \frac{1}{1 + \exp[(E - \mu_\alpha)/k_B T_\alpha]}$$

Mean electric currents and heat currents:

$$J_\alpha = \frac{e}{h} \int_0^\infty dE \sum_\beta \mathcal{T}_{E,\mathbf{B}}^{\alpha\beta} (f_E^\alpha - f_E^\beta)$$

$$Q_\alpha = \frac{1}{h} \int_0^\infty dE \sum_\beta \mathcal{T}_{E,\mathbf{B}}^{\alpha\beta} (E - \mu_\alpha)(f_E^\alpha - f_E^\beta)$$

Electric current fluctuations:

$$P_\alpha = P_\alpha^{\mathrm{th}} + P_\alpha^{\mathrm{sh}}$$

thermal noise:

$$P_\alpha^{\mathrm{th}} = \frac{e^2}{h} \int_0^\infty dE \sum_{\beta \neq \alpha} \mathcal{T}_{E,\mathbf{B}}^{\alpha\beta} \{ f_E^\alpha (1 - f_E^\alpha) + f_E^\beta (1 - f_E^\beta) \}$$

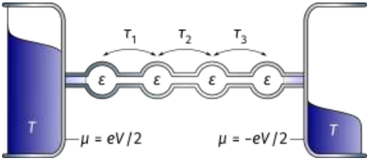
shot noise:

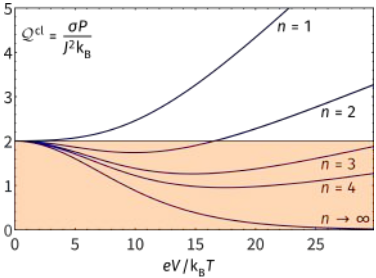
$$P_\alpha^{\mathrm{sh}} = \frac{e^2}{2h} \int_0^\infty dE \sum_{\beta\gamma} \mathcal{T}_{E,\mathbf{B}}^{\alpha\beta} \mathcal{T}_{E,\mathbf{B}}^{\alpha\gamma} (f_E^\beta - f_E^\gamma)^2$$

Unitarity and TR symmetry:

$$\sum_\alpha \mathcal{T}_{E,\mathbf{B}}^{\alpha\beta} = \sum_\beta \mathcal{T}_{E,\mathbf{B}}^{\alpha\beta} \qquad \mathcal{T}_{E,\mathbf{B}}^{\alpha\beta} = \mathcal{T}_{E,-\mathbf{B}}^{\beta\alpha} \geq 0$$

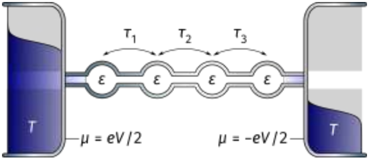
Chain of Quantum Dots

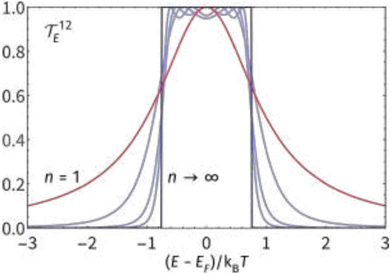


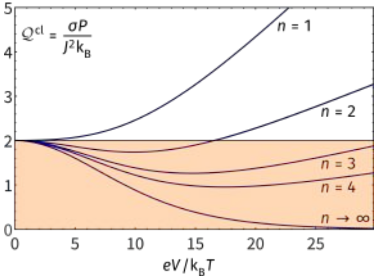


- B. K. Agarwalla, D. Segal, PRB **98** 155438 (2018).
- M. Gerry, D. Segal, PRB **105**, 155401 (2022).

Chain of Quantum Dots







A closer look at current fluctuations:

$$P_\alpha = P_\alpha^{\rm th} + P_\alpha^{\rm sh} = P_\alpha^{\rm cl} - P_\alpha^{\rm ex} + P_\alpha^{\rm sh}$$

quasi-calssical noise:

$$P_\alpha^{\rm cl} = \frac{e^2}{h}\int_0^\infty dE \sum_{\beta\neq\alpha} \mathcal{T}_{E,\mathbf{B}}^{\alpha\beta}\{f_E^\alpha(1-f_E^\beta)+f_E^\beta(1-f_E^\alpha)\} \geq 0$$

exchange correction:

$$P_\alpha^{\rm ex} = \frac{e^2}{h}\int_0^\infty dE \sum_{\beta\neq\alpha} \mathcal{T}_{E,\mathbf{B}}^{\alpha\beta}(f_E^\alpha - f_E^\beta)^2 \geq 0$$

shot noise:

$$P_\alpha^{\rm sh} = \frac{e^2}{2h}\int_0^\infty dE \sum_{\beta\gamma} \mathcal{T}_{E,\mathbf{B}}^{\alpha\beta}\mathcal{T}_{E,\mathbf{B}}^{\alpha\gamma}(f_E^\beta - f_E^\gamma)^2 \geq 0$$

Bound 1:

$$\psi k_{\rm B} \leq \frac{\sigma P_\alpha^{\rm cl}}{J_\alpha^2} \leq \frac{\sigma(P_\alpha + P_\alpha^{\rm ex})}{J_\alpha^2}$$

Bound 2:

$$P_\alpha^{\rm ex} \leq e^2\sigma/2k_{\rm B}$$

TUR for Coherent Transport

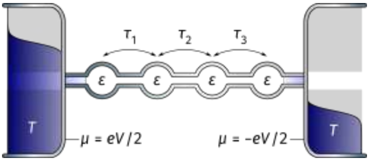
$$\frac{\sigma(P_\alpha + e^2\sigma/2k_{\rm B})}{J_\alpha^2} \geq \psi k_{\rm B} \quad \psi = \begin{cases} 2 & (\mathbf{B}=0) \\ 0.89612 & (\mathbf{B}\neq 0) \end{cases}$$

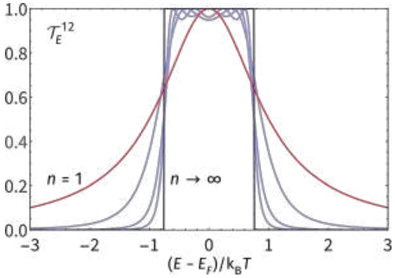
Bound on entropy production:

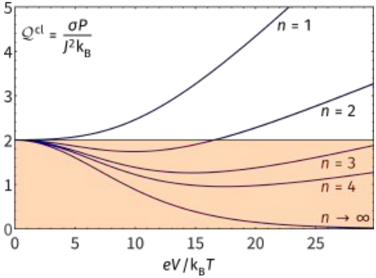
$$\sigma \geq \frac{|J_\alpha| k_{\rm B}}{e}\left(\sqrt{F_\alpha^2 + 2\psi} - |F_\alpha|\right)$$

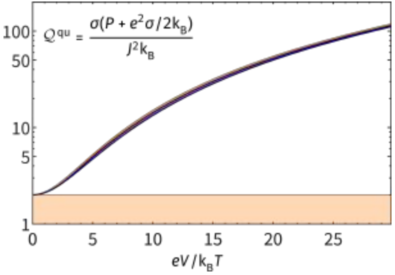
$$F_\alpha = P_\alpha / eJ_\alpha$$

Chain of Quantum Dots



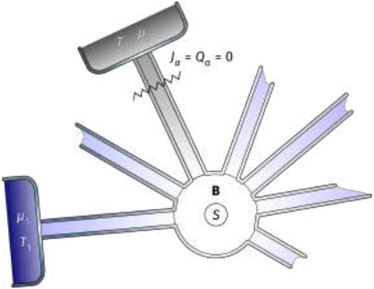






Extensions

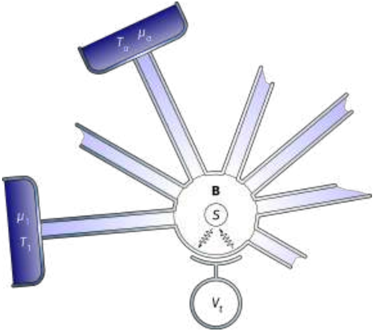
Dephasing and inelastic scattering



TUR

$$\frac{\sigma(P_\alpha + e^2\sigma/2k_B)}{J_\alpha^2} \geq \psi k_B$$

Periodic driving

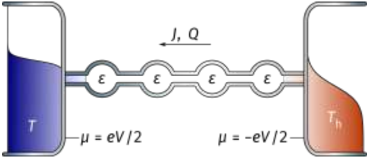


TUR

$$\frac{\sigma(P_\alpha + e^2\sigma/2k_B + 2k_B T_\alpha e^2/h)}{J_\alpha^2} \geq \psi k_B$$

Applications

Thermolectric heat engines



➣ P. Pietzonka, U. Seifert, PRL **120** 190602 (2018).

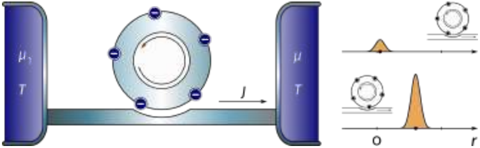
Bound on efficiency:

$$\eta = \frac{\Pi_{\mathrm{el}}}{Q} \leq \frac{\eta_C}{1 + (k_B T/eV)\left(\sqrt{F_{\mathrm{el}}^2 + 2\psi} - F_{\mathrm{el}}\right)}$$

$$\Pi_{\mathrm{el}} = VJ$$

$$F_{\mathrm{el}} = P_{\Pi_{\mathrm{el}}}/eV\Pi_{\mathrm{el}}$$

Quantum Motors



$$\eta = \frac{\Pi_{\mathrm{m}}}{\Pi_{\mathrm{el}}} \leq 1 - \frac{k_B T}{eV}\left(\sqrt{\mathcal{F}_{\mathrm{el}}^2 + 2\psi} - \mathcal{F}_{\mathrm{el}}\right)$$

$$\mathcal{F}_{\mathrm{el}} = \left(P_{\Pi_{\mathrm{el}}} + 2k_B T(eV)^2/h\right)/eV\Pi_{\mathrm{el}}$$

➣ R. Bustos-Marún, G. Refael, F. von Oppen, PRL **111** 060802 (2013).

Conclusions

TUR for Coherent Transport

$$\frac{\sigma(P_\alpha + e^2\sigma/2k_B)}{J_\alpha^2} \geq \psi k_B \quad \psi = \begin{cases} 2 & (\mathbf{B}=0) \\ 0.9 & (\mathbf{B}\neq 0) \end{cases}$$

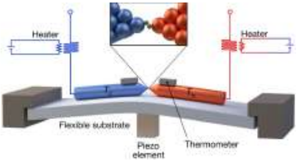
$$\frac{\sigma(P_\alpha + e^2\sigma/2k_B + 2k_B T_\alpha e^2/h)}{J_\alpha^2} \geq \psi k_B$$

Open problems

- Saturation beyond LR
- Heat current fluctuations
- Bosonic carriers

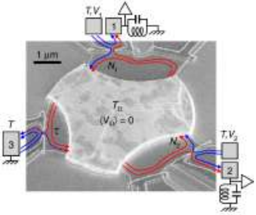
  ➻ S. Saryal et al., PRE **100** 042101 (2019).
- Unifying framework

Atomic junctions



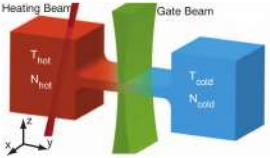
➻ O. S. Lumbroso et al., Nature **562** 240 (2018).

Semiconductors



➻ E. Sivre et al., Nat. Com. **10** 5638 (2019).

Cold atoms



➻ J.-P. Brantut et al., Science **342** 713 (2013).